

LINEARITY DEFECT OF THE RESIDUE FIELD OF SHORT LOCAL RINGS

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ABSTRACT. Let (R, \mathfrak{m}, k) be a Noetherian local ring with maximal ideal \mathfrak{m} and residue field k . The linearity defect of a finitely generated R -module M , which is denoted $\text{ld}_R(M)$, is a numerical measure of how far M is from having linear resolution. We study the linearity defect of the residue field. We give a positive answer to the question raised by Herzog and Iyengar of whether $\text{ld}_R(k) < \infty$ implies $\text{ld}_R(k) = 0$, in the case when $\mathfrak{m}^4 = 0$.

1. INTRODUCTION AND NOTATION

This paper is concerned with the notion of the linearity defect of the residue field of a commutative Noetherian local ring. This invariant was introduced by Herzog and Iyengar [3] and has been further studied by Iyengar and Römer [4], Şega [6] and Nguyen [5]. Let us recall the definition of the linearity defect. Throughout this paper (R, \mathfrak{m}, k) will denote a commutative Noetherian local ring with maximal ideal \mathfrak{m} and residue field k . Let

$$\mathbf{F} : \cdots \rightarrow F_n \xrightarrow{\partial_n} F_{n-1} \xrightarrow{\partial_{n-1}} \cdots \rightarrow F_1 \xrightarrow{\partial_1} F_0 \rightarrow 0$$

be a minimal complex (i.e. $\partial_i(F_i) \subseteq \mathfrak{m}F_{i-1}$ for all $i \geq 0$) of finitely generated free R -modules. Then the complex has a filtration $\{\mathfrak{F}^p \mathbf{F}\}_{p \geq 0}$ with $(\mathfrak{F}^p \mathbf{F})_i = \mathfrak{m}^{p-i} F_i$ for all p and i where, by convention, $\mathfrak{m}^j = R$ for all $j \leq 0$. The associated graded complex with respect to this filtration is called *the linear part* of \mathbf{F} and denoted by $\text{lin}^R(\mathbf{F})$. Let N be an R -module. The notation $R^{\mathfrak{g}}$ will stand for the associated graded ring $\bigoplus_{i \geq 0} \mathfrak{m}^i / \mathfrak{m}^{i+1}$ and $N^{\mathfrak{g}}$ for the associated graded $R^{\mathfrak{g}}$ -module $\bigoplus_{i \geq 0} \mathfrak{m}^i N / \mathfrak{m}^{i+1} N$. By construction, $\text{lin}^R(\mathbf{F})$ is a graded complex of graded free $R^{\mathfrak{g}}$ -modules and has the property that $\text{lin}_n^R(\mathbf{F}) = F_n^{\mathfrak{g}}(-n)$, for all n . For more information about this complex, we again refer to [3] and [4]. Let M be a finitely generated R -module. The *linearity defect* of M is defined to be the number

$$\text{ld}_R(M) := \sup\{i \in \mathbb{Z} \mid H_i(\text{lin}^R(\mathbf{F})) \neq 0\},$$

where \mathbf{F} is a minimal free resolution of M . By definition, $\text{ld}_R(M)$ can be infinite and $\text{ld}_R(M) \leq d$ is finite if and only if $(\text{Syz}_d(M))^{\mathfrak{g}}$ has a linear resolution over the standard graded algebra $R^{\mathfrak{g}}$, where $\text{Syz}_d(M)$ is the d th syzygy module of M . In particular, $\text{ld}_R(M) = 0$ if and only if $M^{\mathfrak{g}}$ has a linear resolution over $R^{\mathfrak{g}}$ and then $\text{lin}^R(\mathbf{F})$ is a minimal graded free resolution of $M^{\mathfrak{g}}$. The notion of the linearity defect can be defined, in the same manner, for graded modules over a standard graded

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algebra A over a field k . In [3], the authors proved that if $\text{ld}_A(k) < \infty$, then $\text{ld}_A(k) = 0$. Motivated by this known result in the graded case, the following natural question raised in [3].

Question 1. If $\text{ld}_R(k) < \infty$, does it follow that $\text{ld}_R(k) = 0$?

If R^g is Cohen-Macaulay, Şega [6] showed that the question has positive answer in the case that R is a complete intersection. Also, she gave an affirmative answer when $\mathfrak{m}^3 = 0$. Another positive answer to the question is given by the author and Rossi [1] when R is of homogeneous type, that is $\dim_k \text{Tor}_i^R(k, k) = \dim_k \text{Tor}_i^{R^g}(k, k)$ for all i .

In this paper we show that this problem has an affirmative answer when $\mathfrak{m}^4 = 0$. The proof relies on the existence of a DG algebra structure of a minimal free resolution of residue field k .

2. PRELIMINARIES AND THE MAIN RESULT

Şega provided an interpretation of linearity defect in term of vanishing of special maps. For each $n \geq 0$ and $i \geq 0$ we consider the map

$$v_i^n(M) : \text{Tor}_i^R(M, R/\mathfrak{m}^{n+1}) \rightarrow \text{Tor}_i^R(M, R/\mathfrak{m}^n)$$

induced by the natural surjection $R/\mathfrak{m}^{n+1} \rightarrow R/\mathfrak{m}^n$. For simplicity, we set $v_i^n := v_i^n(k)$ when $M = k$.

Theorem 2.1. [6, Theorem 2.2] *Let M be a finitely generated R -module and d be an integer. Then the following conditions are equivalent.*

- (1) $\text{ld}_R(M) \leq d$;
- (2) $v_i^n(M) = 0$ for all $i \geq d + 1$ and all $n \geq 0$.

Remark 2.2. Let $i \geq 0$. Assume that \mathbf{F} is a minimal free resolution of a finitely generated R -module M . Then by [6, 2.3 (2')], the following statements are equivalent.

- (1) $v_i^1(M) = 0$;
- (2) if $x \in F_i$ satisfies $\partial_i(x) \in \mathfrak{m}^2 F_{i-1}$, then $x \in \mathfrak{m} F_i$.

Let S be a unitary commutative ring. Given an S -complex \mathbf{C} , we write $|c| = i$ (the homological degree of c) when $c \in C_i$. When we write $c \in \mathbf{C}$ we mean $c \in C_i$ for some i . A (graded commutative) DG algebra over S is a non-negative S -complex (\mathbf{D}, ∂) with a morphism of complexes called the product

$$\begin{aligned} \mu^{\mathbf{D}} : \mathbf{D} \otimes_S \mathbf{D} &\rightarrow \mathbf{D} \\ a \otimes b &\mapsto ab \end{aligned}$$

satisfying the following properties:

- (i) unital: there is an element $1 \in D_0$ such that $1a = a1 = a$ for $a \in \mathbf{D}$;
- (ii) associative: $a(ba) = (ab)c$ for all $a, b, c \in \mathbf{D}$;
- (iii) graded commutative: $ab = (-1)^{|a||b|}ba \in D_{|a|+|b|}$ for all $a, b \in \mathbf{D}$ and $a^2 = 0$ when $|a|$ is odd.

The fact that μ is a morphism of complexes is expressed by the *Leibniz rule*:

$$\partial(ab) = \partial(a)b + (-1)^{|a|}a\partial(b)$$

For more information on DG algebras we refer to [2].

Remark 2.3. If (\mathbf{D}, ∂) is a DG algebra over S . Using Leibniz rule, one can see that the subcomplex of cycles $Z(\mathbf{D})$ is a DG subalgebra of \mathbf{D} and the boundaries $B(\mathbf{D})$ is a DG ideal of $Z(\mathbf{D})$. Thus the product on \mathbf{D} induces a product on the homology $H(\mathbf{D}) = Z(\mathbf{D})/B(\mathbf{D})$. In particular, $\oplus_{i \geq 0} H_n(\mathbf{D})$ is a graded module over commutative ring $H_0(\mathbf{D})$.

Tate constructed a DG algebra (free) resolution of k . Furthermore, such a resolution can be chosen to be minimal, see [2, Theorem 6.3.5], which we refer a minimal Tate resolution of k over R .

The following lemma shows that the linear part of a minimal Tate resolution of k inherits a DG algebra structure from that of the resolution.

Lemma 2.4. *Let (\mathbf{F}, ∂) be a minimal Tate resolution of k . Then $\text{lin}^R(\mathbf{F})$ has a DG algebra structure induced by that of \mathbf{F} .*

Proof. Let $\mu^{\mathbf{F}} : \mathbf{F} \otimes_R \mathbf{F} \rightarrow \mathbf{F}$ be a morphism of complexes which defines the product on \mathbf{F} . Set $S = R^{\mathfrak{g}}$. Since \mathbf{F} is minimal we see that $\mathbf{F} \otimes_R \mathbf{F}$ is a minimal complex as well. Hence the morphism induces a morphism of graded S -complexes $\text{lin}^R(\mu^{\mathbf{F}}) : \text{lin}^R(\mathbf{F} \otimes_R \mathbf{F}) \rightarrow \text{lin}^R(\mathbf{F})$ such that if $i, n \geq 0$ and x^* is the image of an element $x \in \mathfrak{m}^i(\mathbf{F} \otimes_R \mathbf{F})_n$ in $\mathfrak{m}^i(\mathbf{F} \otimes_R \mathbf{F})_n / \mathfrak{m}^{i+1}(\mathbf{F} \otimes_R \mathbf{F})_n$, then $\text{lin}^R(\mu^{\mathbf{F}})$ maps x^* into the image of $\mu(x)$ in $\mathfrak{m}^i F_n / \mathfrak{m}^{i+1} F_n$. There is also a natural isomorphism of graded S -complexes $\lambda : \text{lin}^R(\mathbf{F}) \otimes_S \text{lin}^R(\mathbf{F}) \rightarrow \text{lin}^R(\mathbf{F} \otimes_R \mathbf{F})$ such that if $i, j, n, m \geq 0$ and $x \in \mathfrak{m}^i F_n$ and $y \in \mathfrak{m}^j F_m$ with images x^* in $\mathfrak{m}^i F_n / \mathfrak{m}^{i+1} F_n$ and y^* in $\mathfrak{m}^j F_m / \mathfrak{m}^{j+1} F_m$ respectively, then λ maps $x^* \otimes y^*$ into the image of $x \otimes y$ in $\mathfrak{m}^{i+j}(\mathbf{F} \otimes_R \mathbf{F})_{n+m} / \mathfrak{m}^{i+j+1}(\mathbf{F} \otimes_R \mathbf{F})_{n+m}$, see [4, Lemma 2.7]. Now, define (the product)

$$\mu^{\text{lin}^R(\mathbf{F})} : \text{lin}^R(\mathbf{F}) \otimes_S \text{lin}^R(\mathbf{F}) \rightarrow \text{lin}^R(\mathbf{F})$$

as the composition $\text{lin}^R(\mu^{\mathbf{F}}) \circ \lambda$. Since μ satisfies conditions (i), (ii), (iii) of the definition of DG algebras, one can see that $\mu^{\text{lin}^R(\mathbf{F})}$ satisfies the same properties as well. Therefore the linear part of \mathbf{F} is a DG algebra over S augmented to k . \square

Let \mathfrak{m}^* denote the homogeneous maximal ideal of $R^{\mathfrak{g}}$. The following is a direct consequence of the above lemma.

Corollary 2.5. *If \mathbf{F} is a minimal free resolution of k , then $\mathfrak{m}^* H_n(\text{lin}^R(\mathbf{F})) = 0$ for all n .*

Proof. The assertion follows from Remark 2.3 with considering the fact that $H_0(\text{lin}^R(\mathbf{F})) = R^{\mathfrak{g}} / \mathfrak{m}^*$. \square

In what will follow, let (\mathbf{F}, ∂) be a minimal free resolution of residue field k with differential map ∂ . The differential map of $\text{lin}^R(\mathbf{F})$ which is induced by ∂ will be denoted by ∂^* . We recall that $\text{lin}^R(\mathbf{F})_n = F_n^{\mathfrak{g}}(-n)$. For any $i, n \geq 0$ and $x \in \mathfrak{m}^i F_n$, ∂^* maps $x + \mathfrak{m}^{i+1} F_n$, the image of x in $\mathfrak{m}^i F_n / \mathfrak{m}^{i+1} F_n$, into the image of $\partial(x)$ in $\mathfrak{m}^{i+1} F_{n-1} / \mathfrak{m}^{i+2} F_{n-1}$ that is $\partial(x) + \mathfrak{m}^{i+2} F_{n-1}$.

Proposition 2.6. *Let d be an integer. If $\text{ld}_R(k) \leq d$, then the following hold.*

- (1) $v_d^1 = 0$.
- (2) $\mathfrak{m}^* \text{Ker } \partial_d^* = \mathfrak{m}^* \text{Im } \partial_{d+1}^*$.

Proof. For the simplicity, we set $Z = \text{Ker } \partial_d^*$ and $B = \text{Im } \partial_{d+1}^*$.

(1) If $v_d^1 \neq 0$, then there exists an element $e \in F_d \setminus \mathfrak{m}F_d$ such that $\partial_d(e) \in \mathfrak{m}^2 F_{d-1}$, by 2.2. Let e^* be the image of e in the quotient module $F_d/\mathfrak{m}F_d$. Then $\partial_d^*(e^*) = 0$ and so e^* is a cycle in $\text{lin}^R(\mathbf{F})$. Applying 2.5, we have $\mathfrak{m}^*Z \subseteq B$ and therefore $\mathfrak{m}^*e^* \subseteq B$. As e^* is an element of a basis of the free module $F_d^{\mathfrak{g}}(-d)$ and $B \subseteq \mathfrak{m}^*F_d^{\mathfrak{g}}(-d)$, it is straightforward to see that \mathfrak{m}^*e^* is a direct summand of B . By the hypothesis, B has a linear resolution. This implies that the same property holds for \mathfrak{m}^*e^* . Therefore k has a linear resolution over $R^{\mathfrak{g}}$ and consequently $\text{lin}^R(\mathbf{F})$ is acyclic. Hence $v_d^1 = 0$ and this is a contradiction.

For (2), it is enough to show that $\mathfrak{m}^*Z \subseteq \mathfrak{m}^*B$. First we claim that \mathfrak{m}^*Z is generated in degree at least $d+2$ and $\mathfrak{m}^*Z \subseteq B$. Indeed since $v_d^1 = 0$, applying Remark 2.2, one has $Z \subseteq \mathfrak{m}^*F_d^{\mathfrak{g}}(-d)$ and consequently \mathfrak{m}^*Z is generated in degree at least $d+2$. The second part of the claim follows from Corollary 2.5.

On the other hand, B has a linear resolution, by the hypothesis. Hence B is generated by elements of degree $d+1$ and then all its elements of degree at least $d+2$ contained in \mathfrak{m}^*B . Now, putting these two considerations together, we get $\mathfrak{m}^*Z \subseteq \mathfrak{m}^*B$. \square

Now, we are ready to prove our main result.

Theorem 2.7. *Assume that R is Artinian with $\mathfrak{m}^4 = 0$. If $\text{ld}_R(k) < \infty$, then $\text{ld}_R(k) = 0$.*

Proof. Let d be a non-negative integer and $\text{ld}_R(k) \leq d$. We prove by descending induction on d . The case where $d = 0$ is clear. Let $d > 0$. Applying Proposition 2.6, we have $v_d^1 = 0$. Since $\mathfrak{m}^4 = 0$, it follows from [6, Theorem 7.1] that $v_d^2 = 0$. Again since $\mathfrak{m}^4 = 0$ obviously $v_d^i = 0$ for all $i \geq 3$, by the definition of the map v_d^i . Therefore, from 2.1 we get $\text{ld}_R(k) \leq d-1$. This completes the induction and finishes the proof. \square

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